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NONLINEAR SYSTEMS

Quasi-Optimal Angular Acceleration of a Spacecraft Based on the Poinsot Concept

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Abstract—This paper is devoted to the kinematic problem of the optimal, in terms of time, program angular acceleration of a spacecraft as a solid body under arbitrary (given) boundary conditions imposed on its angular position and angular velocity. A quasi-optimal analytical solution of the problem is obtained within the classical Poinsot concept of the angular motion of a solid body as a generalized coning motion and Pontryagin's maximum principle. This solution is brought to an algorithm. Supporting analytical and numerical examples are provided to show either the proximity of the quasi-optimal and optimal solutions or their complete coincidence, depending on the boundary conditions.

Keywords: optimal and quasi-optimal control, spacecraft, solid body, angular acceleration, Poinsot concept, analytical solution, algorithm

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1. INTRODUCTION

In many spacecraft guidance and control problems, it is required to know the optimal, in some sense, program angular acceleration of a spacecraft under arbitrary (given) boundary conditions [1–5]. In the literature, as a rule, such acceleration is either numerically found or analytically constructed (e.g., [5]) based on polynomials (splines) by representing the spacecraft attitude quaternion via polynomials and expressing the angular velocity vector through this quaternion. However, no guarantees (rigorously proved theorems or considerations from theoretical mechanics) are given that these analytical solutions will approximate the optimal angular motion trajectory of the spacecraft well enough on the entire set of its angular motions under any boundary conditions.

In this paper, we study the kinematic problem of the optimal, in terms of time, program angular acceleration of a spacecraft under arbitrary boundary conditions for its angular position and angular velocity. It is difficult to obtain the optimal analytical solution of this problem under arbitrary boundary conditions: in the general case, the exact solution of the attitude problem of a solid body by its known angular velocity (the Darboux problem) remains unknown [6, 7]. Following the classical Poinsot concept of the angular motion of a solid body as a generalized coning motion, we reformulate the optimal angular acceleration problem of a spacecraft in this class of motions. The spacecraft trajectory is given by explicit expressions containing arbitrary quaternionic and scalar constants and two arbitrary scalar functions (the parameters of the generalized coning motion). These parameters and their derivatives are employed as variables to pose and solve an optimization problem in which the second derivatives of the parameters act as controls. Note that the generality of the original problem is almost not violated: the known exact solution of the classical optimal angular acceleration problem in the planar rotation case [4] or in the new special case of regular precession (considered below) and the similar solutions of the problem within the above concept completely coincide; in other cases, in numerical calculations of the solution of the original problem and the analytical solution proposed, the relative error between the values of an optimization criterion (the determinative characteristic of the problem) constitutes at most 1%, including spacecraft rotations to large angles. Therefore, the analytical solution proposed in this paper can be treated as quasi-optimal with respect to that of the classical optimal angular acceleration problem of a spacecraft. Explicit expressions for the attitude quaternion and the angular velocity vector of a spacecraft are derived. An explicit formula for the angular acceleration of a spacecraft is obtained by differentiating the expression for the angular velocity vector. Finally, an analytical algorithm for solving the problem is presented, which can be used aboard the spacecraft.

The analytical solution method proposed below has previously been successfully applied to dynamic optimal rotation problems for spacecraft of various configurations with different optimization criteria [8, 9].

2. CLASSICAL PROBLEM FORMULATION

The angular motion of a spacecraft as a solid body around its center of gravity is described by a system of differential equations in vector-matrix form:

$$2\begin{bmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & -\lambda_0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \tag{2.1}$$

$$\begin{bmatrix} \dot{\omega_1} & \dot{\omega_2} & \dot{\omega_3} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}^{\mathrm{T}}, \tag{2.2}$$

which is equivalent to the short representation [4]

$$2\dot{\mathbf{\Lambda}} = \mathbf{\Lambda} \circ \boldsymbol{\omega}, \tag{2.3}$$

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon}, \tag{2.4}$$

where $\mathbf{\Lambda} = [\lambda_0(t), \lambda_1(t), \lambda_2(t), \lambda_3(t)]^{\mathrm{T}}$ or $\lambda_0(t)+\lambda_1(t)i_1+\lambda_2(t)i_2+\lambda_3(t)i_3$ is the normalized quaternion that describes the angular position of the spacecraft ($||\mathbf{\Lambda}|| = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_0^2 = 1$); $\boldsymbol{\omega} = [\omega_1(t), \omega_2(t), \omega_3(t)]^{\mathrm{T}}$ or $\boldsymbol{\omega}(t) = \omega_1(t)\mathbf{i}_1+\omega_2(t)\mathbf{i}_2+\omega_3(t)\mathbf{i}_3$ is the angular velocity vector of the spacecraft, which can be treated as a quaternion with the zero scalar part $\boldsymbol{\omega} = [0, \omega_1(t), \omega_2(t), \omega_3(t)]^{\mathrm{T}}$; $\boldsymbol{\varepsilon} = [\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)]^{\mathrm{T}}$ is the angular acceleration vector of the spacecraft; the imaginary units i_1, i_2, and i_3 of the Hamiltonian correspond to the vectors of the orthonormal basis of the three-dimensional vector space $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively; $\circ$ indicates the quaternion product; finally, $^{\mathrm{T}}$ means the vector transpose. Accordingly, the phase coordinates $\mathbf{\Lambda}$ and $\boldsymbol{\omega}$ are assumed to be continuous functions, and the angular acceleration $\boldsymbol{\varepsilon}$ (control) is assumed to be a piecewise continuous function [10].

The angular acceleration vector is bounded by magnitude:

$$|\ \boldsymbol{\varepsilon}\ | \leqslant \varepsilon_{\max}. \tag{2.5}$$

The boundary conditions imposed on the attitude and angular velocity of the spacecraft are arbitrary:

$$\mathbf{\Lambda}(0) = \mathbf{\Lambda}_0, \quad \mathbf{\Lambda}(T) = \mathbf{\Lambda}_T, \tag{2.6}$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \quad \boldsymbol{\omega}(T) = \boldsymbol{\omega}_T. \tag{2.7}$$

It is required to find the optimal control $\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)$ that minimizes the criterion

$$J = T. \tag{2.8}$$

The problem formulation (2.1)–(2.8) is kinematic since it neglects the dynamic configuration of the spacecraft. Note that the optimal angular acceleration of the spacecraft can be found as the derivative of its optimal angular velocity in the full dynamic optimal spacecraft attitude problem, where the Euler dynamic equations are used instead of the vector equation (2.4) and the control function is the vector of the external control moment applied to the spacecraft [8, 9].

3. DIMENSIONLESS VARIABLES

We reformulate the problem by replacing the original dimensional variables with dimensionless ones:

$$t^{\mathrm{dimless}} = t\varepsilon_{\max}^{1/2}, \quad \boldsymbol{\omega}^{\mathrm{dimless}} = \boldsymbol{\omega}\varepsilon_{\max}^{1/2}, \quad \boldsymbol{\varepsilon}^{\mathrm{dimless}} = \boldsymbol{\varepsilon}\varepsilon_{\max}^{1/2}.$$

In this case, all the expressions will not change, except for the magnitude constraint on the angular acceleration vector:

$$|\,\boldsymbol{\varepsilon}\,| \leqslant 1. \tag{3.1}$$

Below we will solve problem (2.3), (2.4), (2.6)–(2.8), (3.1) in the dimensionless variables, with the removed superscripts in the problem formulation.

4. APPLICATION OF THE MAXIMUM PRINCIPLE

Applying Pontryagin's maximum principle [4, 10], we introduce two variables $\boldsymbol{\Psi}(t)$ and $\boldsymbol{\varphi}(t)$, quaternion and vector, respectively, that are conjugate to the phase variables. The Hamilton–Pontryagin function is given by

$$H = (\boldsymbol{\Psi},\, \boldsymbol{\Lambda} \circ \boldsymbol{\omega})/2 + (\boldsymbol{\varphi},\, \boldsymbol{\varepsilon}), \tag{4.1}$$

where (. , .) means the inner product of vectors.

The conjugate system of equations has the form

$$\begin{cases} 2\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \circ \boldsymbol{\omega}, \\ \dot{\boldsymbol{\varphi}} = -\mathrm{vect}\left(\tilde{\boldsymbol{\Lambda}} \circ \boldsymbol{\Psi}\right)/2, \end{cases} \tag{4.2}$$

where vect(.) and ˜ are the vector part and conjugation of the quaternion, respectively. The linear differential systems of equations for the variables $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ coincide, hence their solutions differ by a quaternion constant $\mathbf{C}$:

$$\boldsymbol{\Psi} = \mathbf{C} \circ \boldsymbol{\Lambda}. \tag{4.3}$$

The transversality condition [10] at the time instant $t = T$ is

$$\mathrm{scal}\,(\boldsymbol{\Psi}(T) \circ \tilde{\boldsymbol{\Lambda}}(T)) = 0, \tag{4.4}$$

where scal(.) indicates the scalar part of the quaternion. The expression scal $(\boldsymbol{\Psi} \circ \tilde{\boldsymbol{\Lambda}})$ is the first integral for the system of equations (4.2). Therefore, in view of (4.4),

$$\mathrm{scal}\,(\boldsymbol{\Psi}(t) \circ \tilde{\boldsymbol{\Lambda}}(t)) \equiv 0. \tag{4.5}$$

From (4.3) and (4.5) it follows that scal $\mathbf{C} = 0$ and $\mathbf{C} = \mathbf{c}_v$, where $\mathbf{c}_v$ is a constant vector. Then equality (4.3) takes the form

$$\boldsymbol{\Psi} = \mathbf{c}_v \circ \boldsymbol{\Lambda}. \tag{4.6}$$

By denoting [4]

$$\mathbf{p} = \text{vect} \ \left(\tilde{\boldsymbol{\Lambda}} \circ \boldsymbol{\Psi}\right) = \tilde{\boldsymbol{\Lambda}} \circ \mathbf{c}_v \circ \boldsymbol{\Lambda}, \tag{4.7}$$

we write the system of equations (4.2) as

$$\begin{cases} \mathbf{p} = \tilde{\boldsymbol{\Lambda}} \circ \mathbf{c}_v \circ \boldsymbol{\Lambda}, \\ \dot{\boldsymbol{\varphi}} = -\mathbf{p}/2. \end{cases} \tag{4.8}$$

Thus, due to the self-conjugation of the linear differential system of equations (2.1) ((2.3)), the dimension of the boundary value problem of the maximum principle is reduced by four [4].

The maximum condition for the Hamilton–Pontryagin function (4.1) on the bounded and closed set (3.1) yields the following structure of optimal control:

$$\boldsymbol{\varepsilon}^{\text{opt}} = \boldsymbol{\varphi}/|\boldsymbol{\varphi}|. \tag{4.9}$$

Note that the case of special control $\boldsymbol{\varphi} \equiv 0$ violates the requirement of the maximum principle regarding the existence of nontrivial conjugate variables $\boldsymbol{\varphi}$ and $\boldsymbol{\Psi}$ [10]. Indeed, formula (4.8) implies $\mathbf{p} \equiv 0$ and $\mathbf{c}_v \equiv 0$; formula (4.6), $\boldsymbol{\Psi} \equiv 0$.

According to (2.3), (2.4), (4.8), and (4.9), we obtain

$$\dot{\mathbf{p}} = [\mathbf{p}, \boldsymbol{\omega}], \tag{4.10}$$

$$\mathbf{p} = -2|\boldsymbol{\varphi}|\frac{d^2\boldsymbol{\omega}}{dt^2} - 2\frac{d\boldsymbol{\omega}}{dt}\frac{d|\boldsymbol{\varphi}|}{dt}, \tag{4.11}$$

where $[.\,,.]$ means the cross product of vectors.

Substituting the expression (4.11) into equation (4.10) gives

$$\frac{d^3\boldsymbol{\omega}}{dt^3} = \left[\frac{d^2\boldsymbol{\omega}}{dt^2}, \boldsymbol{\omega}\right] + \frac{1}{|\boldsymbol{\varphi}|}\left(\frac{d|\boldsymbol{\varphi}|}{dt}\left(\left[\frac{d\boldsymbol{\omega}}{dt}, \boldsymbol{\omega}\right] - 2\frac{d^2\boldsymbol{\omega}}{dt^2}\right) - \frac{d^2|\boldsymbol{\varphi}|}{dt^2}\frac{d\boldsymbol{\omega}}{dt}\right). \tag{4.12}$$

The optimal angular velocity of the spacecraft on the entire time interval of its motion satisfies the third-order vector differential equation (4.12). (Previously, a similar fact was established for special control regimes in the optimal turn problem of a spherically symmetric spacecraft [11].) Based on (4.7), the Hamilton–Pontryagin function (4.1) takes the form

$$H = (\mathbf{p}, \boldsymbol{\omega})/2 + (\boldsymbol{\varphi}, \boldsymbol{\varepsilon}]). \tag{4.13}$$

5. THE OPTIMAL ANGULAR ACCELERATION PROBLEM OF SPACECRAFT: THE EXACT SOLUTION IN THE SPECIAL CASE

Now we present a new partial solution of the optimal angular acceleration problem of the spacecraft in the class of regular coning motions. For this purpose, let the magnitude of the vector $\boldsymbol{\varphi}$ be constant on the entire time interval of spacecraft motion:

$$|\ \boldsymbol{\varphi}\ | = c. \tag{5.1}$$

In this case, the conjugate variables $\mathbf{p}$ and $\boldsymbol{\varphi}$ and the control $\boldsymbol{\varepsilon}$ are expressed through the angular velocity vector by the formulas

$$\boldsymbol{\varphi} = c\dot{\boldsymbol{\omega}}, \quad \mathbf{p} = -2c\dot{\boldsymbol{\omega}}, \quad \boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}, \tag{5.2}$$

and the angular velocity vector satisfies the differential equation

$$\dddot{\boldsymbol{\omega}} = [\ddot{\boldsymbol{\omega}}, \boldsymbol{\omega}]\,. \tag{5.3}$$

Thus, the optimal control problem (2.3), (2.4), (2.6)–(2.8), (3.1) is reduced to the boundary value problem (2.3), (5.3), (2.6), (2.7). Without loss of generality, assume that $c = 1$.

The optimal angular velocity of the spacecraft in the class of regular coning motions has the form

$$\boldsymbol{\omega} = \widetilde{\mathbf{K}} \circ (\mathbf{i}_1 \alpha \sin \Omega t + \mathbf{i}_2 \alpha \cos \Omega t + \mathbf{i}_3 \Omega) \circ \mathbf{K}, \tag{5.4}$$

where $\mathbf{K}$ (quaternion), α, and Ω are arbitrary constants; in addition,

$$||\mathbf{K}|| = \mathrm{K}_0^2 + \mathrm{K}_1^2 + \mathrm{K}_2^2 + \mathrm{K}_3^2 = 1. \tag{5.5}$$

The quaternion $\mathbf{K}$ is responsible for the rotation of the parenthesized vector in formula (5.4) about some constant axis passing through a fixed point of the spacecraft. Let us show that the angular velocity (5.4) satisfies equation (5.3). To this end, it is necessary to differentiate formula (5.4) with respect to the variable t sequentially three times: $\dot{\boldsymbol{\omega}} = \alpha\Omega \widetilde{\mathbf{K}} \circ (\mathbf{i}_1 \cos \Omega t - \mathbf{i}_2 \sin \Omega t) \circ \mathbf{K}$, $\ddot{\boldsymbol{\omega}} = -\alpha\Omega^2 \widetilde{\mathbf{K}} \circ (\mathbf{i}_1 \sin \Omega t + \mathbf{i}_2 \cos \Omega t) \circ \mathbf{K}$, $\dddot{\boldsymbol{\omega}} = \alpha\Omega^3 \widetilde{\mathbf{K}} \circ (-\mathbf{i}_1 \cos \Omega t + \mathbf{i}_2 \sin \Omega t) \circ \mathbf{K}$. Substituting (5.4) and the resulting expressions for the derivatives into (5.3), we arrive at the desired equality. In addition, $\dddot{\boldsymbol{\omega}} = [\ddot{\boldsymbol{\omega}}, \boldsymbol{\omega}] = (\ddot{\boldsymbol{\omega}} \circ \boldsymbol{\omega} - \boldsymbol{\omega} \circ \ddot{\boldsymbol{\omega}})/2$.

The trajectory of the spacecraft with the angular velocity (5.4) is found explicitly from (2.3) and (2.6). It represents a regular precession of the form

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{K}} \circ \mathbf{exp}\{\mathbf{i}_2 \alpha t/2\} \circ \mathbf{exp}\{\mathbf{i}_3 \Omega t/2\} \circ \mathbf{K}, \tag{5.6}$$

where $\mathbf{exp}\{.\}$ denotes the exponential function with a quaternionic index [4].

The expressions (5.4)–(5.6) include five arbitrary constants α, Ω, and K_i, $i = 0, 1, 2$. (The constant K_3 is related to condition (5.5).) The magnitude constraint on the angular acceleration vector $\boldsymbol{\varepsilon}$ is true under the condition

$$\alpha^2 \Omega^2 = 1. \tag{5.7}$$

Let us satisfy the boundary conditions (2.6) and (2.7). Due to the insufficient number of arbitrary constants in (5.4), we impose requirements on $|\boldsymbol{\omega}_0|$ and $\boldsymbol{\omega}_T$ during the problem solution, with the unit vector $\boldsymbol{\omega}_0^e = \boldsymbol{\omega}_0 / |\boldsymbol{\omega}_0|$ being arbitrary. At the point $t = 0$, by formula (5.4),

$$\boldsymbol{\omega}_0 = |\boldsymbol{\omega}_0|\, \boldsymbol{\omega}_0^e = \widetilde{\mathbf{K}} \circ (\mathbf{i}_2 \alpha + \mathbf{i}_3 \Omega) \circ \mathbf{K}, \tag{5.8}$$

$$\|\boldsymbol{\omega}_0\| = \left\| \tilde{\mathbf{K}} \circ (\mathbf{i}_2 \alpha + \mathbf{i}_3 \Omega) \circ \mathbf{K} \right\| = \left\| \widetilde{\mathbf{K}} \right\| \|\mathbf{i}_2 \alpha + \mathbf{i}_3 \Omega\| \, \|\mathbf{K}\| = \alpha^2 + \Omega^2; \tag{5.9}$$

at the right end of the spacecraft trajectory (for $t = T$), from (2.6) and (5.6) we have

$$\boldsymbol{\Lambda}_T = \boldsymbol{\Lambda}_0 \circ \tilde{\mathbf{K}} \circ \mathbf{exp}\{\mathbf{i}_2 \alpha T/2\} \circ \mathbf{exp}\{\mathbf{i}_3 \Omega T/2\} \circ \mathbf{K}, \tag{5.10}$$

and

$$\mathrm{scal}(\tilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T) = \mathrm{scal}(\mathbf{exp}\{\mathbf{i}_2 \alpha T/2\} \circ \mathbf{exp}\{\mathbf{i}_3 \Omega T/2\}). \tag{5.11}$$

The values of T, $|\boldsymbol{\omega}_0|$, α, Ω, and $\mathbf{K}$ can be determined using formulas (5.5) and (5.7)–(5.10). Let (5.8) and (5.10) be written as

$$(\mathbf{i}_2\alpha + \mathbf{i}_3\Omega)\circ\mathbf{K} - \mathbf{K}\circ\boldsymbol{\omega}_0 = 0,$$
$$\mathbf{exp}\{\mathbf{i}_2\alpha/2\}\circ\mathbf{exp}\{\mathbf{i}_3\Omega/2\}\circ\mathbf{K} - \mathbf{K}\circ\tilde{\boldsymbol{\Lambda}}_0\circ\boldsymbol{\Lambda}_T = 0.$$

The same representation based on $m-$ and $n-$matrices isomorphic to quaternions [12] yields

$$\left(\begin{bmatrix} 0 & 0 & -\alpha & -\Omega \\ 0 & 0 & -\Omega & \alpha \\ \alpha & \Omega & 0 & 0 \\ \Omega & -\alpha & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\omega_{01} & -\omega_{02} & -\omega_{03} \\ \omega_{01} & 0 & \omega_{03} & -\omega_{02} \\ \omega_{02} & -\omega_{03} & 0 & \omega_{01} \\ \omega_{03} & \omega_{02} & -\omega_{01} & 0 \end{bmatrix}\right)\begin{pmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.12}$$

$$\left(\begin{bmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ m_1 & m_0 & -m_3 & m_2 \\ m_2 & m_3 & m_0 & -m_1 \\ m_3 & -m_2 & m_1 & m_0 \end{bmatrix} - \begin{bmatrix} n_0 & -n_1 & -n_2 & -n_3 \\ n_1 & n_0 & n_3 & -n_2 \\ n_2 & -n_3 & n_0 & n_1 \\ n_3 & n_2 & -n_1 & n_0 \end{bmatrix}\right)\begin{pmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.13}$$

where the coefficient matrix of the linear algebraic system (5.13) is defined by the components of the quaternions $\mathbf{m}$ and $\mathbf{n}$:

$$\begin{cases} \mathbf{m} = \mathbf{exp}\{\mathbf{i}_2\alpha/2\}\circ\mathbf{exp}\{\mathbf{i}_3\Omega/2\}, \\ m_0 = \cos(\alpha/2)\cos(\Omega/2), \quad m_1 = \sin(\alpha/2)\sin(\Omega/2), \\ m_2 = \sin(\alpha/2)\cos(\Omega/2), \quad m_3 = \cos(\alpha/2)\sin(\Omega/2), \end{cases} \tag{5.14}$$

$$\mathbf{n} = \tilde{\boldsymbol{\Lambda}}_0\circ\boldsymbol{\Lambda}_T. \tag{5.15}$$

Their norms are $\|\mathbf{m}\| = 1$ and $\|\mathbf{n}\| = 1$, and the coefficient matrices of systems (5.12) and (5.13) have rank 2. By choosing linearly independent equations, two in (5.12) and two in (5.13), we obtain the homogeneous system

$$\begin{bmatrix} 0 & \omega_{01} & \omega_{02}-\alpha & \omega_{03}-\Omega \\ -\omega_{01} & 0 & -(\omega_{03}+\Omega) & \omega_{02}+\alpha \\ m_1-n_1 & 0 & -(m_3+n_3) & m_2+n_2 \\ 0 & n_1-m_1 & n_2-m_2 & n_3-m_3 \end{bmatrix}\begin{pmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.16}$$

For the existence of nontrivial solutions, the determinant of the coefficient matrix of system (5.16) must be 0. Proceeding from this requirement and (5.8), (5.14), and (5.15), we obtain

$$|\boldsymbol{\omega}_0| = (m_2\alpha + m_3\Omega)/(n_1\omega_{01}^e + n_2\omega_{02}^e + n_3\omega_{03}^e). \tag{5.17}$$

Due to (5.7)–(5.9), (5.11), and (5.17), α, Ω, and T can be determined from the system of three equations

$$\begin{cases} \alpha^2\Omega^2 - 1 = 0, \\ (\alpha^2+\Omega^2)\left(n_1\omega_{01}^e + n_2\omega_{02}^e + n_3\omega_{03}^e\right)^2 - (m_2\alpha + m_3\Omega)^2 = 0, \\ \mathrm{scal}(\tilde{\boldsymbol{\Lambda}}_0\circ\boldsymbol{\Lambda}_T) - \cos(\alpha T/2)\cos(\Omega T/2) = 0. \end{cases} \tag{5.18}$$

The components of the quaternion $\mathbf{K}$ are given by

$$\begin{aligned} &K_3 = \pm\left[1 + (A_0/D)^2 + (A_1/D)^2 + (A_2/D)^2\right]^{-1/2}, \quad K_0 = A_0K_3/D, \\ &K_1 = A_1K_3/D, \qquad\qquad\qquad\qquad\qquad\qquad\qquad K_2 = A_2K_3/D, \end{aligned} \tag{5.19}$$

where

$$\begin{cases} A_0 = -(m_2+n_2)(\omega_{03}+\Omega)+(m_3+n_3)(\omega_{02}+\alpha), \\ A_1 = (m_1-n_1)\omega_{01}+(m_3+n_3)(\Omega-\omega_{03})+(m_2+n_2)(\alpha-\omega_{02}), \\ A_2 = (m_1-n_1)(\omega_{02}+\alpha)+(m_2+n_2)\omega_{01}, \\ D = (m_1-n_1)(\omega_{03}+\Omega)+(m_3+n_3)\omega_{01}. \end{cases} \tag{5.20}$$

The boundary condition for the angular velocity of the spacecraft at $t = T$ takes the form

$$\boldsymbol{\omega}(T) = \boldsymbol{\omega}_T = \widetilde{\mathbf{K}} \circ (\mathbf{i}_1\alpha\sin\Omega T + \mathbf{i}_2\alpha\cos\Omega T + \mathbf{i}_3\Omega) \circ \mathbf{K}. \tag{5.21}$$

Well, if the boundary conditions for the angular velocity $\boldsymbol{\omega}$ of the spacecraft satisfy the requirements (5.17) and (5.21), its angular motion trajectory will be in the class of regular coning motions and given by the explicit expressions (5.4) and (5.6). (For all $t \in [0,\, T]$, $\boldsymbol{\omega}(t)$ will belong to a conic surface defined by arbitrary boundary conditions on the position, $\boldsymbol{\Lambda}_0$ and $\boldsymbol{\Lambda}_T$, and an arbitrary direction of its initial vector $\boldsymbol{\omega}_0^e$.)

According to (2.4) and (5.4), the optimal angular acceleration is

$$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \alpha\Omega\widetilde{\mathbf{K}} \circ (\mathbf{i}_1\cos\Omega t - \mathbf{i}_2\sin\Omega t) \circ \mathbf{K}, \tag{5.22}$$

$$|\boldsymbol{\varepsilon}|^2 = \alpha^2\Omega^2 = 1. \tag{5.23}$$

In other words, the magnitude constraint (3.1) on the control vector holds.

The conjugate variables $\boldsymbol{\varphi}$ and $\mathbf{p}$ are calculated from the expressions (5.2). Thus, the problem with the above constraints has been completely solved.

Now we write an algorithm for finding the optimal angular acceleration of the spacecraft in the class of regular coning motions:

1) Use the known values $\boldsymbol{\Lambda}_0$ and $\boldsymbol{\Lambda}_T$ from (2.6), $\boldsymbol{\omega}_0^e = \boldsymbol{\omega}_0/\left|\boldsymbol{\omega}_0\right|$ from (2.7), (5.17), (5.18), and formulas (5.14) and (5.15) to find the unknowns α, Ω, T, and $\left|\boldsymbol{\omega}_0\right|$.

2) Use formulas (5.19) and (5.20) and the values of $\boldsymbol{\Lambda}_0$, $\boldsymbol{\Lambda}_T$, α, Ω, T, and $|\boldsymbol{\omega}_0|$ to find the quaternion $\mathbf{K}$.

3) Use formulas $\boldsymbol{\omega}_0^{\mathrm{cal}} = |\boldsymbol{\omega}_0|\,\boldsymbol{\omega}_0^e$ and $\boldsymbol{\omega}_T^{\mathrm{cal}} = \widetilde{\mathbf{K}} \circ (\mathbf{i}_1\alpha\sin\Omega T + \mathbf{i}_2\alpha\cos\Omega T + \mathbf{i}_3\Omega) \circ \mathbf{K}$ to find the boundary vectors $\boldsymbol{\omega}_0^{\mathrm{cal}}$ and $\boldsymbol{\omega}_T^{\mathrm{cal}}$.

4) Compare the calculated values $\boldsymbol{\omega}_0^{\mathrm{cal}}$ and $\boldsymbol{\omega}_T^{\mathrm{cal}}$ with the given boundary conditions (2.6).

5) If equality holds in item 4), then the optimal solution of the problem is in the class of regular coning motions; the angular velocity of the spacecraft, its trajectory, angular acceleration vector, and the performance time T are calculated by formulas (5.4), (5.6), and (5.22) and item 1).

6) The conjugate variables $\boldsymbol{\varphi}$ and $\mathbf{p}$ are calculated by formulas (5.2).

This optimal solution of the problem in the class of regular coning motions and the previously known exact solution [4] in the class of flat Euler turns (under the condition $\boldsymbol{\omega}_0, \boldsymbol{\omega}_T\, ||\mathrm{vect}(\widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T)$) for the special cases of boundary conditions on the angular velocity of the spacecraft will be used as analytical confirmations when constructing a quasi-optimal solution of the angular acceleration problem of a spacecraft in the class of generalized coning motions. Also, we will provide some suggestive considerations obtained through the numerical solution of the optimal angular acceleration problem of a spacecraft under arbitrary boundary conditions.

6. JUSTIFICATION OF THE APPROACH BASED ON NUMERICAL SOLUTIONS OF THE PROBLEM

Let us numerically solve the boundary value problem of the maximum principle (Section 3) for the original kinematic optimal acceleration problem of a spacecraft, see (2.3), (2.4), (2.6)–(2.8), and (3.1):

$$\begin{cases} 2\dot{\mathbf{\Lambda}} = \mathbf{\Lambda} \circ \boldsymbol{\omega}, \\ \dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon}, \\ \dot{\boldsymbol{\varphi}} = -\mathbf{p}/2, \\ \mathbf{p} = \tilde{\mathbf{\Lambda}} \circ \mathbf{c}_v \circ \mathbf{\Lambda}, \quad \mathbf{c}_v = \mathbf{const}, \end{cases} \tag{6.1}$$

$$\mathbf{\Lambda}(0) = \mathbf{\Lambda}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \tag{6.2}$$

$$\mathbf{\Lambda}(T) = \mathbf{\Lambda}_T, \quad \boldsymbol{\omega}(T) = \boldsymbol{\omega}_T, \tag{6.3}$$

$$\boldsymbol{\varepsilon}^{\text{opt}} = \boldsymbol{\varphi}/|\boldsymbol{\varphi}|, \tag{6.4}$$

$$H^{\text{opt}}(T) = (\mathbf{p}, \boldsymbol{\omega})/2 + (\boldsymbol{\varphi}, \boldsymbol{\varepsilon})|_{t=T} = 1. \tag{6.5}$$

It is required to find $\boldsymbol{\varepsilon}^{\text{opt}}$, T^{opt}, $\mathbf{\Lambda}^{\text{opt}}$, $\boldsymbol{\omega}^{\text{opt}}$, and $\mathbf{c}_v$.

It seems reasonable to write the terminal conditions (6.3) in the phase space $\mathbf{\Lambda} \times \boldsymbol{\omega}$ of dimension 7 [8, 9, 11]:

$$\text{vect}\ (\mathbf{\Lambda}(T) \circ \tilde{\mathbf{\Lambda}}_T) = 0, \quad \boldsymbol{\omega}(T) = \boldsymbol{\omega}_T. \tag{6.6}$$

A numerical solution method for such problems was described in [8, 9, 11]. Also, for comparison, we provide kinematic characteristics based on the calculation results in the full dynamic optimal rotation problem of a spacecraft as a solid body of various dynamic configurations [8] in dimensionless variables under the same boundary conditions for its angular position and angular velocity:

$$\mathbf{\Lambda}_0 = (0.7951,\ 0.2981,\ -0.3975,\ 0.3478), \quad \boldsymbol{\omega}_0 = (0.2739,\ -0.2388,\ -0.3), \tag{6.7}$$

$$\mathbf{\Lambda}_T = (0.8443,\ 0.3984,\ -0.3260,\ 0.1485), \quad \boldsymbol{\omega}_T = (0.0,\ 0.0,\ -0.59). \tag{6.8}$$

Spacecraft 1. An arbitrary spacecraft, $I_1 = 0.9869$, $I_2 = 1.1843$, and $I_3 = 0.7895$.

Spacecraft 2. $I_1 = 0.9506$, $I_2 = 1.3308$, and $I_3 = 0.5704$.

Spacecraft 3. The International Space Station (ISS) in its early version [13], $I_1 = 4\,853\,000$ kg $\times$ m^2, $I_2 = 23\,601\,000$ kg $\times$ m^2, and $I_3 = 26\,278\,000$ kg $\times$ m^2 or the dimensionless variables $I_1 = 0.2358$, $I_2 = 1.1466$, and $I_3 = 1.2766$.

Spacecraft 4. Space Shuttle, which has characteristics almost like a dynamically symmetric solid body: $I_1 = 3\,400\,648$ kg $\times$ m^2, $I_2 = 21\,041\,672$ kg $\times$ m^2 or $I_1 = 0.1967$, $I_2 = 1.2168$, and $I_3 \approx I_2$.

Table 1 shows the values of the attitude quaternion and angular velocity vector of the spacecraft obtained by solving the optimal angular acceleration problem (6.1)–(6.6) and the dynamic optimal rotation problem [8] for the four spacecraft at the intermediate points $t_* = 0.4508$, $t_1 = 0.4504$, $t_2 = 0.4502$, $t_3 = 0.4513$, and $t_4 = 0.4515$ and the time intervals of their motion $[0,\ T^{\text{opt}}]$, $[0,\ T_k{}^{\text{opt}}]$, $k = 1, 2, 3, 4$ (the corresponding optimal reorientation times are $T^{\text{opt}} = 0.8965$, $T_1{}^{\text{opt}} = 0.8858$,

Table 1

Spacecraft	λ_0	λ_1	λ_2	λ_3	ω_1	ω_2	ω_3
Spacecraft 1 (t_1)	0.80480	0.36812	−0.37874	0.27082	−0.03497	−0.03416	−0.57245
Spacecraft 2 (t_2)	0.80582	0.36894	−0.37740	0.26853	−0.03943	−0.03158	−0.60068
Spacecraft 3 (t_3)	0.80405	0.36949	−0.38005	0.26935	−0.04073	−0.04253	−0.58311
Spacecraft 4 (t_4)	0.80311	0.37231	−0.37955	0.27171	−0.03412	−0.03576	−0.55873
Values in problem (6.1)–(6.6) (t_*)	0.80437	0.36758	−0.37955	0.27171	−0.03412	−0.03576	−0.55873
Values in the modified problem	0.80452	0.36763	−0.37817	0.27309	−0.03246	−0.03050	−0.55603

Table 2

Spacecraft	λ_0	λ_1	λ_2	λ_3	ω_1	ω_2	ω_3
Spacecraft 3 ($t_3{}^0$)	0.82396	0.35523	−0.36235	0.25220	0.02834	−0.05283	−0.61018
Spacecraft 4 ($t_4{}^0$)	0.82698	0.35296	−0.36222	0.24560	0.02813	−0.04666	−0.61162
Values in problem (6.1)–(6.6) (t_*^0)	0.83121	0.36278	−0.35560	0.22588	0.02843	−0.04552	−0.60910
Values in the modified problem	0.83228	0.36140	−0.35278	0.22855	0.02987	−0.05517	−0.60373

$T_2^{\text{opt}} = 0.8654$, $T_3^{\text{opt}} = 0.8774$, and $T_4^{\text{opt}} = 0.8728$.) The intermediate points t_*, t_k are as close to each other as the calculation program with a variable step t allows; they are oriented approximately at the middle of the largest time interval of motion $[0,\ T^{\text{opt}}]$ in the kinematic problem (6.1)–(6.6). For comparison, the last line of Table 1 contains the data obtained by solving the modified angular acceleration problem of a spacecraft (at the point $t = \tau_* = 0.4491$). This problem will be discussed in Sections 7 and 8 of the paper.

Table 2 presents the values of the spacecraft phase coordinates obtained by solving the same problems for spacecraft 3 and 4 in the case of rotation from the rest position to the rest position

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}_T = (0.0,\ 0.0,\ 0.0) \tag{6.9}$$

and the initial and terminal attitude conditions given by the expressions (6.7) and (6.8); the optimal reorientation times are $T^{\text{opt}} = 1.3916$, $T_3^{\text{opt}} = 1.5645$, and $T_4^{\text{opt}} = 1.5278$.

The intermediate points $t_*{}^0$ are approximately close to each other:

$$t_*{}^0 = 0.7809, \quad t_3{}^0 = 0.7823, \quad t_4{}^0 = 0.7815.$$

The same calculations were carried out for other boundary conditions. According to Tables 1 and 2 and other calculations, in the dynamic optimal rotation problem, the phase coordinates of the spacecraft depend significantly on its initial and terminal states, less significantly on its configuration, and are close enough to the results of the kinematic optimal angular acceleration problem of the spacecraft. Note that in the energy-optimal spacecraft rotation problem with fixed time [9], the above effect is more obvious since the time interval of spacecraft motion and the corresponding average time are the same in all examples. Therefore, the kinematic optimal angular acceleration problem (6.1)–(6.6) has a general nature for spacecraft of arbitrary configurations. In this case, expressions for the attitude quaternion and angular velocity in the kinematic problem can be derived analytically in explicit form based on the solution of the modified optimal reorientation problem of the spacecraft in the class of generalized coning motions, and the control angular acceleration can be determined by differentiating the angular velocity vector. We consider this in more detail.

7. THE MODIFIED OPTIMAL ANGULAR ACCELERATION PROBLEM OF SPACECRAFT

The general solution of the fundamental attitude problem of a solid body by its known angular velocity (2.1), (2.3), called the Darboux problem, is unknown. Therefore, we obtain a solution in the class of generalized coning motions. For this purpose, let the angular velocity vector $\boldsymbol{\omega}(t)$ be defined by

$$\boldsymbol{\omega}(t) = \mathbf{i}_1 \dot{f}(t) \sin g(t) + \mathbf{i}_2 \dot{f}(t) \cos g(t) + \mathbf{i}_3 \dot{g}(t), \tag{7.1}$$

where the functions $f(t)$ and $g(t)$ (the parameters of the generalized coning motion) are arbitrary. In this case, equation (2.3) has an exact solution [8, 9] satisfying the initial condition (2.5):

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}_0 \circ \mathbf{exp}\{-\mathbf{i}_3 g(0)/2\} \circ \mathbf{exp}\{-\mathbf{i}_2 f(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2 f(t)/2\} \circ \mathbf{exp}\{\mathbf{i}_3 g(t)/2\}. \tag{7.2}$$

Clearly, the expressions (7.1) and (7.2) include those for the angular velocity and trajectory of the spacecraft in exact solutions of the optimal angular acceleration problem when the angular velocity vector keeps a constant direction on the entire time interval of spacecraft motion [4] or makes a regular precession (see Section 5 of this paper). The Darboux problem can generally be reduced, using bijective changes of variables [8, 9], to an equation of the form (2.3), where the angular velocity is like (7.1) but has the opposite direction

$$\boldsymbol{\omega}^*(t) = -\boldsymbol{\omega}(t)$$

and belongs to the class of similar motions. (The quaternionic differential equation (2.3) with the angular velocity $\boldsymbol{\omega}^*(t)$ still has no explicit solution.) Thus, the form of the angular velocity vector (7.1) corresponds to the classical Poinsot concept of the angular motion of a solid body as a generalized coning motion [7].

By introducing an arbitrary constant quaternion $\mathbf{K}$, $\|\mathbf{K}\| = 1$, into formulas (7.1) and (7.2), we further generalize them:

$$\boldsymbol{\omega} = \widetilde{\mathbf{K}} \circ (\mathbf{i}_1 \dot{f}(t) \sin g(t) + \mathbf{i}_2 \dot{f}(t) \cos g(t) + \mathbf{i}_3 \dot{g}(t)) \circ \mathbf{K}, \tag{7.3}$$

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{K}} \circ \mathbf{exp}\{-\mathbf{i}_3 g(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(f(t) - f(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3 g(t)/2\} \circ \mathbf{K}. \tag{7.4}$$

Let the second derivatives of the functions f and g be treated as controls. With the notation

$$\dot{f} = f_1, \;\; \dot{g} = g_1, \tag{7.5}$$

we compile the controlled system

$$\dot{f} = f_1, \;\; \dot{g} = g_1, \;\; \dot{f}_1 = u_1, \;\; \dot{g}_1 = u_2, \tag{7.6}$$

where f, f_1, g, and g_1 are the phase coordinates of this problem and u_1 and u_2 are the controls. The quaternion $\mathbf{K}$ is defined by

$$\mathbf{K} = \mathbf{K}_2 \circ \mathbf{K}_1, \;\; \mathbf{K}_1 = \mathbf{exp}\{\mathbf{i}_1 \alpha_1/2\}, \;\; \mathbf{K}_2 = \mathbf{exp}\{\mathbf{i}_2 \alpha_2/2\}, \tag{7.7}$$

where α_1 and α_2 are arbitrary constants. Note that the quaternions $\mathbf{K}_1$ and $\mathbf{K}_2$ rotate the angular velocity vector $\boldsymbol{\omega}$ (7.1) about the axes $\mathbf{i}_1$ and $\mathbf{i}_2$, respectively. Due to the additive constant included in the function $g(t)$, the rotation about the axis $\mathbf{i}_3$ is already incorporated in the expressions (7.1) and (7.3). The conjugation of the quaternion has the form

$$\widetilde{\mathbf{K}} = \widetilde{\mathbf{K}}_1 \circ \widetilde{\mathbf{K}}_2, \;\; \widetilde{\mathbf{K}}_1 = \mathbf{exp}\{-\mathbf{i}_1 \alpha_1/2\}, \;\; \widetilde{\mathbf{K}}_2 = \mathbf{exp}\{-\mathbf{i}_2 \alpha_2/2\}. \tag{7.8}$$

In view of (7.7) and (7.8), the functions $\boldsymbol{\omega}$, $\boldsymbol{\Lambda}$ (7.3), (7.4) will satisfy the boundary conditions (2.6), (2.7) if

$$\tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2 \circ (\mathbf{i}_1 f_1(0) \sin g(0) + \mathbf{i}_2 f_1(0) \cos g(0) + \mathbf{i}_3 g_1(0)) \circ \mathbf{K}_2 \circ \mathbf{K}_1 = \boldsymbol{\omega}_0, \tag{7.9}$$

$$\tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2 \circ (\mathbf{i}_1 f_1(T) \sin g(T) + \mathbf{i}_2 f_1(T) \cos g(T) + \mathbf{i}_3 g_1(T)) \circ \mathbf{K}_2 \circ \mathbf{K}_1 = \boldsymbol{\omega}_T, \tag{7.10}$$

$$\begin{gathered} \boldsymbol{\Lambda}_0 \circ \tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2 \circ \mathbf{exp}\{-\mathbf{i}_3 g(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(f(T) - f(0))/2\} \\ \circ \mathbf{exp}\{\mathbf{i}_3 g(T)/2\} \circ \mathbf{K}_2 \circ \mathbf{K}_1 = \boldsymbol{\Lambda}_T. \end{gathered} \tag{7.11}$$

The control angular acceleration of the spacecraft corresponding to the solution of the modified problem is determined by differentiation from equation (2.4). Due to formulas (7.1), (7.5), and (7.6), we obtain

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon} = \widetilde{\mathbf{K}} \circ (\mathbf{i}_1(u_1 \sin g + f_1 g_1 \cos g) + \mathbf{i}_2(u_1 \cos g - f_1 g_1 \sin g) + \mathbf{i}_3 u_2) \circ \mathbf{K}. \tag{7.12}$$

Considering (7.5), (7.7), and (7.8), the components of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ from (7.3), (7.12) take an explicit form:

$$\begin{aligned}
\omega_1 &= f_1 \sin g \cos\alpha_2 - g_1 \sin\alpha_2,\\
\omega_2 &= f_1(\sin g \sin\alpha_1 \sin\alpha_2 - \cos g \cos\alpha_1) + g_1 \sin\alpha_1 \cos\alpha_2,\\
\omega_3 &= f_1(\sin g \cos\alpha_1 \sin\alpha_2 - \cos g \sin\alpha_1) + g_1 \cos\alpha_1 \cos\alpha_2,\\
\varepsilon_1 &= u_1(\sin g + f_1 g_1 \cos g)\cos\alpha_2 - u_2 \sin\alpha_2,\\
\varepsilon_2 &= u_1(\sin g \sin\alpha_1 \sin\alpha_2 + \cos g \cos\alpha_1)\\
&\quad + f_1 g_1(\cos g \sin\alpha_1 \sin\alpha_2 - \sin g \cos\alpha_1) + u_2 \sin\alpha_1 \cos\alpha_2,\\
\varepsilon_3 &= u_1(\sin g \cos\alpha_1 \sin\alpha_2 - \cos g \sin\alpha_1)\\
&\quad + f_1 g_1(\cos g \cos\alpha_1 \sin\alpha_2 + \sin g \sin\alpha_1) + u_2 \cos\alpha_1 \cos\alpha_2.
\end{aligned}$$

The magnitude constraint on the angular acceleration vector of the spacecraft (3.1), (7.12) is given by

$$|\boldsymbol{\varepsilon}| = |u_1^2 + f_1^2 g_1^2 + u_2^2| \leqslant 1. \tag{7.13}$$

To satisfy condition (7.13), we impose the following requirement on the controls u_1 and u_2:

$$\sqrt{u_1^2 + u_2^2} \leqslant u_*, \tag{7.14}$$

where the constant u_* $(0 < u_* < 1)$ is specified by meeting condition (7.13).

For the controlled system (7.6), the optimization problem is posed as follows: find optimal controls $u_1(t)$ and $u_2(t)$ transferring system (7.6) from the state

$$f = f(0), \quad f_1 = f_1(0), \quad g = g(0), \quad g_1 = g_1(0) \tag{7.15}$$

to the state

$$f = f(T), \quad f_1 = f_1(T), \quad g = g(T), \quad g_1 = g_1(T) \tag{7.16}$$

that satisfy the relations (7.9)–(7.11), where the parameters α_1 and α_2 are to be determined, and implement the control objective (optimality criterion)

$$J = T \longrightarrow \min. \tag{7.17}$$

We write the expressions (7.9)–(7.11) as

$$(\mathbf{i}_1 f_1(0)\sin g(0) + \mathbf{i}_2 f_1(0)\cos g(0) + \mathbf{i}_3 g_1(0)) = \mathbf{K}_2 \circ \mathbf{K}_1 \circ \boldsymbol{\omega}_0 \circ \tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2, \tag{7.18}$$

$$(\mathbf{i}_1 f_1(T)\sin g(T) + \mathbf{i}_2 f_1(T)\cos g(T) + \mathbf{i}_3 g_1(T)) = \mathbf{K}_2 \circ \mathbf{K}_1 \boldsymbol{\omega}_T \circ \tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2, \tag{7.19}$$

$$\begin{aligned}
&\mathbf{exp}\{-\mathbf{i}_3 g(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(f(T) - f(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3 g(T)/2\}\\
&\qquad = \mathbf{K}_2 \circ \mathbf{K}_1 \circ \tilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T \circ \tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2.
\end{aligned} \tag{7.20}$$

This problem will be called the modified optimal angular acceleration problem of a spacecraft, and its exact solution is acceptable as an approximate or quasi-optimal solution of the classical optimal problem (2.3)–(2.8), (3.1) ((6.1)–(6.6)).

8. THE SOLUTION OF THE MODIFIED OPTIMAL ANGULAR ACCELERATION PROBLEM OF SPACECRAFT

The Hamilton–Pontryagin function of problem (7.6) is given by

$$H = -1 + \psi_1 f_1 + \psi_2 g_1 + \psi_3 u_1 + \psi_4 u_2, \tag{8.1}$$

where the conjugate variables satisfy the differential system

$$\dot{\psi}_1 = 0, \quad \dot{\psi}_2 = 0, \quad \dot{\psi}_3 = -\psi_1, \quad \dot{\psi}_4 = -\psi_2. \tag{8.2}$$

The general solution of system (8.2) has the form

$$\psi_1 = c_1, \quad \psi_2 = c_2, \quad \psi_3 = -c_1 t + c_3, \quad \psi_4 = -c_2 t + c_4, \tag{8.3}$$

where $c_1, \ldots, c_4$ are arbitrary constants.

The maximum condition for the Hamilton–Pontryagin function (8.1) on the bounded and closed set (7.14) yields the following expressions for the optimal controls:

$$\begin{aligned} u_1 &= u_*(-c_1 t + c_3)/\sqrt{(-c_1 t + c_3)^2 + (-c_2 t + c_4)^2}, \\ u_2 &= u_*(-c_2 t + c_4)/\sqrt{(-c_1 t + c_3)^2 + (-c_2 t + c_4)^2}. \end{aligned} \tag{8.4}$$

By substituting formulas (8.4) into equations (7.6), we find their general solution containing 8 unknown constants $c_1, \ldots, c_8$:

$$\begin{aligned} f =& - c_1 u_* \{t/2 - A/2 - Bc_2/c_1)F(t) \\ &+ B[B/2 + (t - A)c_2/c_1] \ln(t - A + F(t))\}/C + c_5 t + c_6, \\ g =& - c_2 u_* \{t/2 - A/2 - Bc_1/c_2)F(t) \\ &+ B[-B/2 + (t - A)c_1/c_2] \ln(t - A + F(t))\}/C + c_7 t + c_8, \\ f_1 =& - c_1 u_*[F(t) + B \ln(t - A + F(t))c_2/c_1]/C + c_5, \\ g_1 =& - c_2 u_*[F(t) + B \ln(t - A + F(t))c_1/c_2]/C + c_7, \\ A =& (c_1 c_3 + c_2 c_4)/C^2, \quad B = (c_1 c_4 - c_2 c_3)/C^2, \quad C = \sqrt{c_1^2 + c_2^2}, \\ F(t) =& \sqrt{(t - A)^2 + B^2}. \end{aligned} \tag{8.5}$$

Based on formula (7.4), let $c_6 = 0$ in the expression for the function f (8.5). The nine unknown constants of the problem, $c_1, \ldots, c_5,\ c_7,\ c_8, \alpha_1$, and α_2, and the time T are determined from the nine equations of system (7.18)–(7.20) and the condition that the Hamilton–Pontryagin function (8.1) equals 0 at the finite time instant. (Due to the requirement $||\mathbf{\Lambda}|| = 1$, three scalar equations are independent in the quaternionic representation (7.20).) By using (7.3), (7.4), and (8.5), we obtain explicit dependencies determining the laws of change of the angular velocity and trajectory of the spacecraft in the optimal angular acceleration problem in the class of generalized coning motions. In view of (8.4) and (8.5), formula (7.12) gives an analytical expression for the angular acceleration vector $\boldsymbol{\varepsilon}$. Thus, the modified optimal angular acceleration problem of the spacecraft has been completely solved.

If $c_1 c_4 - c_2 c_3 \neq 0$, the controls u_1 and u_2 and the phase coordinates $f,\ f_1,\ g$, and g_1 (8.4), (8.5) correspond to the continuous controls u_1 and u_2 and, by formula (7.12), to the continuous control angular acceleration $\boldsymbol{\varepsilon}$. In the case $c_1 c_4 - c_2 c_3 = 0$, first-order discontinuities of the control functions u_1, u_2 and angular acceleration $\boldsymbol{\varepsilon}$ are allowed, and the constants c_1, c_2, c_3, and c_4 obey the condition

$$c_3/c_1 = c_4/c_2 = t_*. \tag{8.6}$$

In this case, the control functions (8.4) become piecewise continuous:

$$u_1 = -c_1 u_* \operatorname{sgn}(t - t_*)/C, \quad u_2 = -c_2 u_* \operatorname{sgn}(t - t_*)/C. \tag{8.7}$$

If $t_* \in [0, T]$, then by (8.7) we have two-stage control with constant sections or, otherwise, one-stage constant control. Based on (7.6) and (8.7), it is possible to express the phase coordinates as polynomials of degree 1 and 2 of the variable t :

$$\begin{aligned}
f &= -c_1 u_*(t - t_*)^2 \operatorname{sgn}(t - t_*)/2C + c_5(t - t_*) + c_6, \\
g &= -c_2 u_*(t - t_*)^2 \operatorname{sgn}(t - t_*)/2C + c_7(t - t_*) + c_8, \\
f_1 &= -c_1 u_*(t - t_*)^2 \operatorname{sgn}(t - t_*)/2C + c_5 = c_1 u_* |t - t_*|/C + c_5, \\
g_1 &= -c_2 u_*(t - t_*)^2 \operatorname{sgn}(t - t_*)/2C + c_7 = c_2 u_* |t - t_*|/C + c_7,
\end{aligned} \tag{8.8}$$

where f, f_1, g, and g_1 are continuous at the control discontinuity point $t = t_*$.

Note that $c_6 = 0$ and $c_1 = \pm 1$. (The expressions (8.7) and (8.8) can be reformulated when c_2, c_3, and c_4 appear as ratios to c_1.) Then (8.7) and (8.8) will contain 5 arbitrary constants c_2, c_5, c_7, c_8, and t_*. The nine scalar equations (7.18)–(7.20) serve to determine them and the unknowns T, α_1, and α_2. Hence, solutions of the problem with the piecewise constant controls u_1 and u_2 arise under a definite relationship between the given values of the boundary conditions of problem (2.6), (2.7). Varying the values of c_2, c_5, c_7, c_8, T, α_1, and α_2 of equations (7.18)–(7.20) yields a set of values of $\boldsymbol{\omega}_0$, $\boldsymbol{\Lambda}_T$, and $\boldsymbol{\omega}_T$ for which the solution of the optimal angular acceleration problem of the spacecraft will be obtained under a discontinuity of the controls u_1 and u_2 and, consequently, under a discontinuity of the angular acceleration $\boldsymbol{\varepsilon}$.

If the vectors $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_T$ of the angular velocity boundary conditions in the optimal angular acceleration problem of the spacecraft are collinear to the vector $\operatorname{vect}(\tilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T)$ (the case of the planar Euler rotation of the spacecraft [4]), the solutions of the original and modified problems will coincide. The same is true when the solution of the original problem is found in the class of regular coning motions (see Section 5 of the paper). In this case, $f_1^2 g_1^2 = 0$ in (7.13), and the magnitude of the angular acceleration vector of the spacecraft in the modified problem equals the constraint value (consequently, is constant).

In the optimal angular acceleration problem of a spacecraft from (4.9), the magnitude of the angular acceleration vector is $|\boldsymbol{\varepsilon}| \equiv 1$; in the modified problem, from (8.4) it follows that $\sqrt{u_1^2 + u_2^2} = u_*$, where u_* is determined from the requirement that the angular acceleration calculated satisfies the condition $|\boldsymbol{\varepsilon}| \leqslant 1$. The angular acceleration obtained by solving the modified problem may fail to satisfy the condition $|\boldsymbol{\varepsilon}| \equiv 1$. Because of this, the performance time in the modified problem may slightly differ from that in the classical problem.

Now we write an algorithm for finding the optimal angular acceleration of the spacecraft under arbitrary boundary conditions.

1. In view of (8.3)–(8.5), use the boundary conditions $\boldsymbol{\Lambda}_0$, $\boldsymbol{\Lambda}_T$ (2.6), $\boldsymbol{\omega}_0$, $\boldsymbol{\omega}_T$ (2.7), formulas (7.7), (7.8), the nine scalar equations from (7.18)–(7.20), and the equality $H(T) = 0$ (8.1) to find the nine arbitrary constants $c_1, \ldots, c_5$, c_7, c_8, α_1, and α_2 and the time T; then determine $f(t)$, $f_1(t)$, $g(t)$, and $g_1(t)$.

2. Calculate the quaternion $\mathbf{K}$ by formula (7.7).

3. Find the law of change of the angular velocity of the spacecraft by (7.3):

$$\boldsymbol{\omega} = \widetilde{\mathbf{K}} \circ (\mathbf{i}_1 \dot{f}(t) \sin g(t) + \mathbf{i}_2 \dot{f}(t) \cos g(t) + \mathbf{i}_3 \dot{g}(t)) \circ \mathbf{K}.$$

4. Find the law of change of the angular motion trajectory of the spacecraft by (7.4):

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{K}} \circ \mathbf{exp}\{-\mathbf{i}_3 g(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2 (f(t) - f(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3 g(t)/2\} \circ \mathbf{K}.$$

5. Construct the angular velocity vector of the spacecraft by formula (7.12).

9. NUMERICAL EXAMPLES

In this section, we compare the numerical solution results of the original (classical) optimal angular acceleration problem of a spacecraft and the quasi-optimal solution of this problem using the analytical algorithm from Section 8. The values of α_1, α_2, $c_1, \dots, c_5$, c_7, c_8, and u_* in the quasi-optimal solution of the spacecraft rotation problem with the boundary conditions (6.7), (6.8) are given by

$$\alpha_1 = -0.04218, \quad \alpha_2 = -0.22280, \quad c_1 = 1.0, \quad c_2 = -1.03798,$$
$$c_3 = 0.67161 \quad c_4 = -0.70520, \quad c_5 = -0.01875,$$
$$c_7 = -0.73653, \quad c_8 = -0.73932, \quad u_* = 0.99450.$$

The values of the quasi-optimal and optimal performance times are $T^{\text{quasi-opt}} = 0.8982$ and $T^{\text{opt}} = 0.8965$ (Section 6); their difference is $\Delta T = T^{\text{quasi-opt}} - T^{\text{opt}} = 0.0017$, and the relative error makes up $(\Delta T/T^{\text{opt}}) \times 100\% = 0.19\%$.

Figures 1 and 2 show the graphs of the projections of the spacecraft angular velocity vector $\omega_i(t)$, the components of the spacecraft attitude quaternion $\Lambda_i(t)$, and the projections of the spacecraft angular acceleration vector $\varepsilon_i(t)$, $i = 1, 2, 3$, depending on the time variable t; the graphs are quite similar for the two problems. Next, Table 3 contains the values of the projections of the spacecraft angular acceleration vector $\boldsymbol{\varepsilon}(t)$ when solving the classical ($\boldsymbol{\varepsilon}^{\text{opt}}$) and modified ($\boldsymbol{\varepsilon}^{\text{quasi-opt}}$) problems at the ends and intermediate points, $t_* = 0.4508$ and $\tau_* = 0.4491$, of the time intervals of spacecraft motion in the two solutions.

Table 3

t	$\varepsilon_1^{\text{opt}}$	$\varepsilon_2^{\text{opt}}$	$\varepsilon_3^{\text{opt}}$	t	$\varepsilon_1^{\text{quasi-opt}}$	$\varepsilon_2^{\text{quasi-opt}}$	$\varepsilon_3^{\text{quasi-opt}}$
0	−0.65603	0.48884	−0.57503	0	−0.63762	0.51267	−0.57285
t_*	−0.70530	0.41389	−0.57554	τ_*	−0.72143	0.38795	−0.56710
T^{opt}	0.78625	−0.29616	0.54230	$T^{\text{quasi-opt}}$	0.84240	−0.07923	0.52813

The calculations for different boundary conditions show the closeness of the solutions of the classical and modified angular acceleration problems of a spacecraft. Therefore, the solution of the modified problem can be treated as a quasi-optimal solution of the classical optimal angular acceleration problem of a spacecraft.

As an example, we also demonstrate the results of solving the kinematic optimal angular acceleration problem of a spacecraft in the class of regular coning motions using the analytical algorithm from Section 5. The graphs in Fig. 3 are the projections of the variables $\boldsymbol{\omega}(t)$, $\boldsymbol{\Lambda}(t)$, and $\boldsymbol{\varepsilon}(t)$ under the boundary conditions with $\boldsymbol{\Lambda}_0 = (0.9975,\ 0.0504,\ 0.0504,\ 0)$, $\boldsymbol{\Lambda}_T = (0.5495,\ 0.4582,\ 0.2291,\ 0.6598)$, $\boldsymbol{\omega}_0 = (1.1811,\ 0.5906,\ 0.7382)$, $\boldsymbol{\omega}_T = (0.0093,\ 0.6860,\ 1.3447)$, which satisfy the requirements of Section 5. First, based on (5.17)–(5.20), the constants α_1, Ω, T, $|\boldsymbol{\omega}_0|$, K_0, K_1, K_2, and K_3 were calculated ($\alpha_1 = 1.3041$, $\Omega = -0.7668$, $T = 1.2967$, $|\boldsymbol{\omega}_0| = 1.5129$, $K_0 = 0.5810$, $K_1 = -0.6561$, $K_2 = -0.3122$, and $K_3 = 0.3669$); then the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ and the quaternion $\boldsymbol{\Lambda}$ were constructed from (5.4), (5.22), and (5.6).

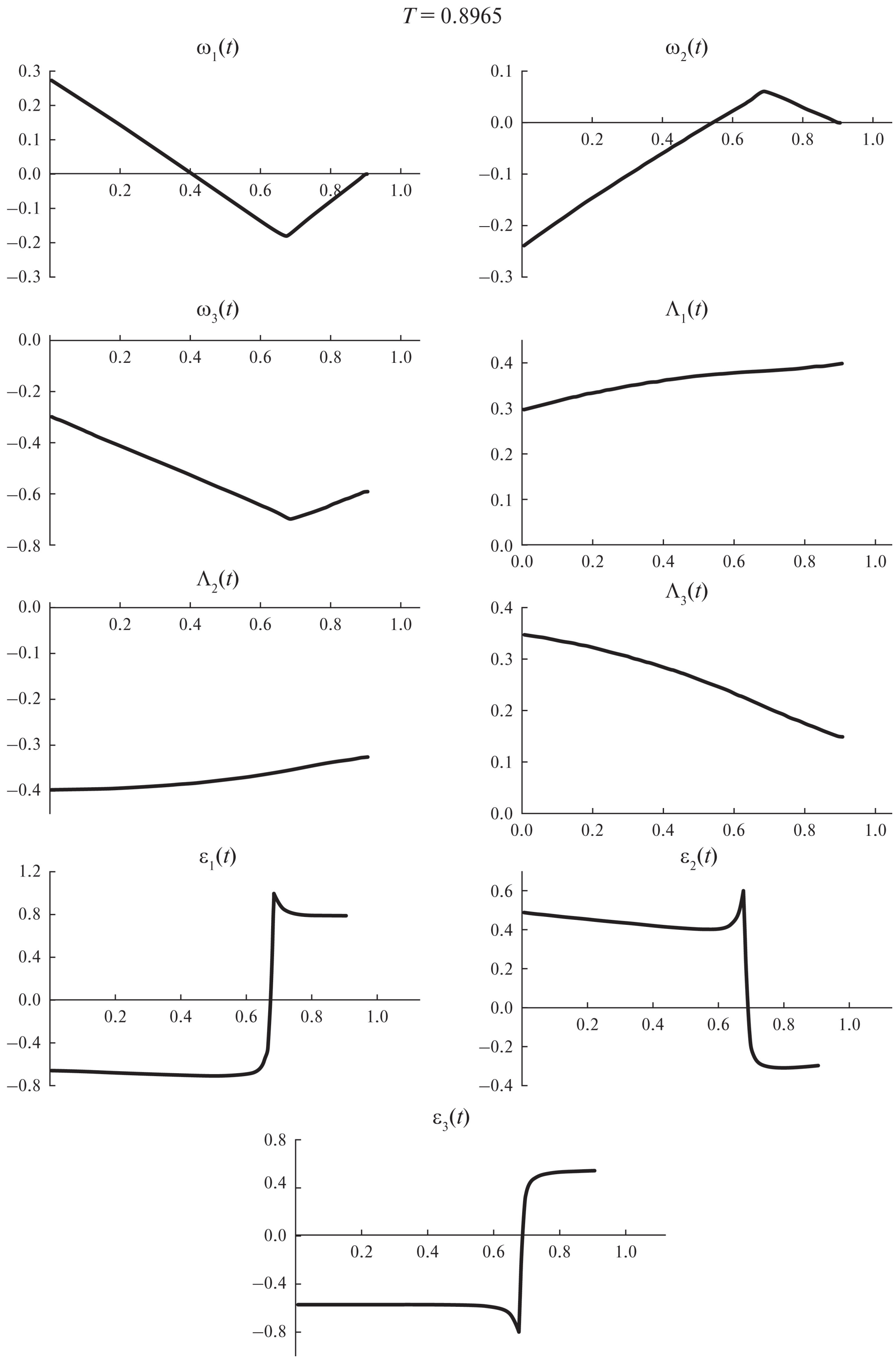

Fig. 1. The solution of the classical problem under arbitrary boundary conditions.

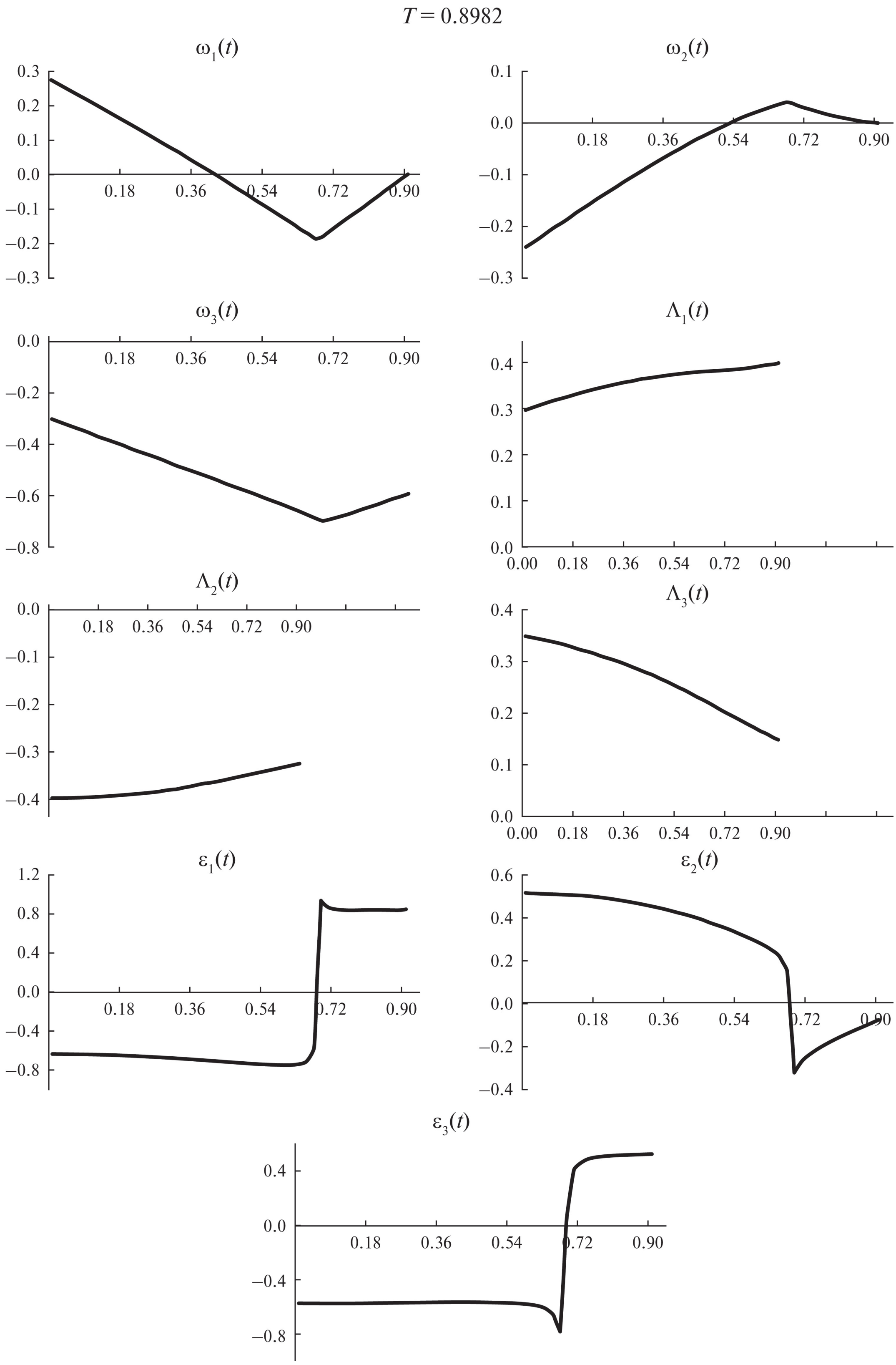

Fig. 2. The solution of the modified problem under arbitrary boundary conditions.

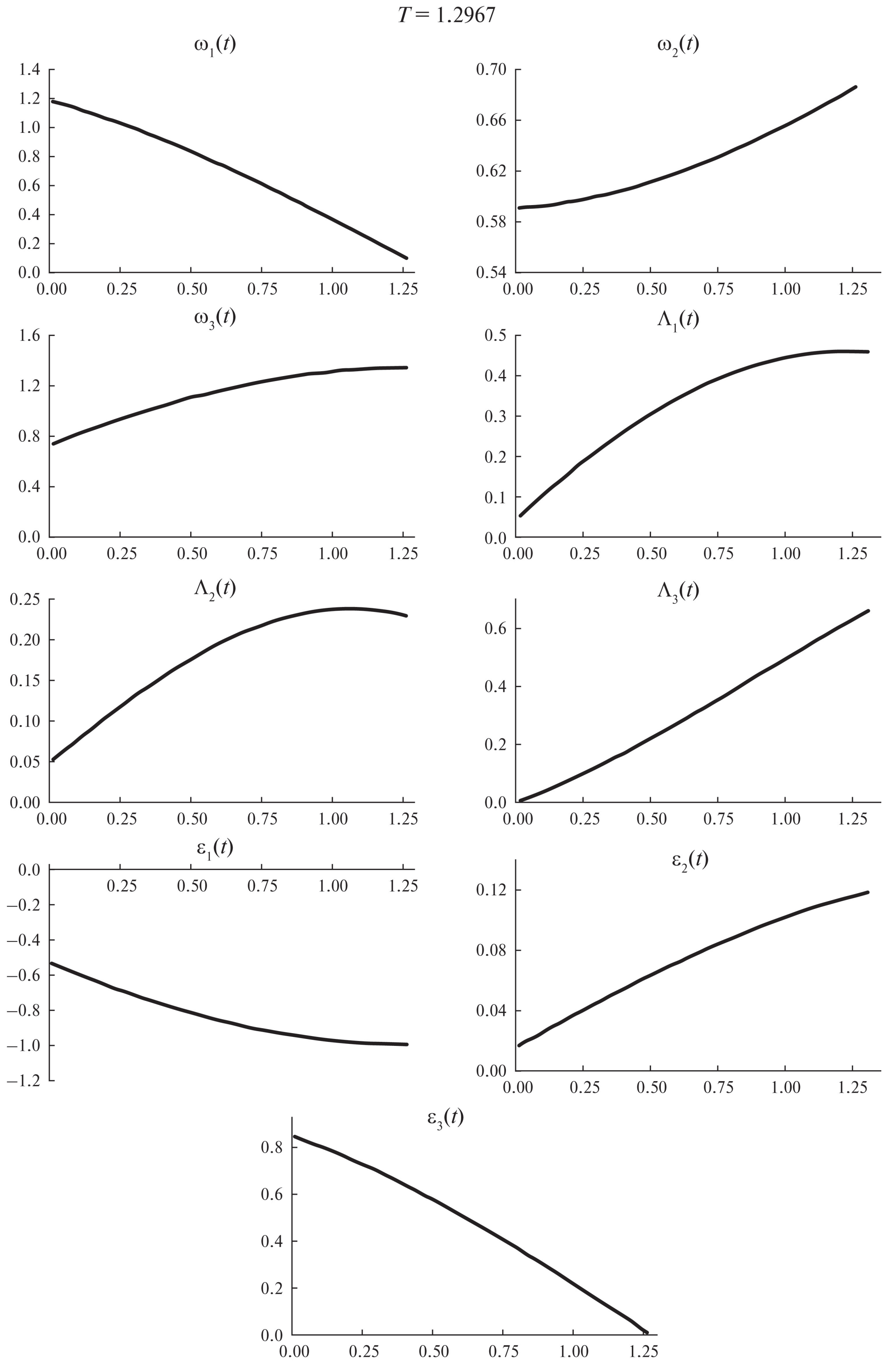

Fig. 3. The solution of the classical problem in the case of regular coning motion.

10. CONCLUSIONS

The analytical quasi-optimal algorithm for solving the kinematic optimal program angular acceleration problem of a spacecraft under arbitrary boundary conditions is theoretically justified and yields the optimal kinematic angular acceleration of a spacecraft with acceptable accuracy. This algorithm does not require a numerical solution of the boundary value problem of the maximum principle or other complex numerical solution; it provides ready-made analytical laws of quasi-optimal program control and program trajectory change that can be installed aboard a spacecraft. In this regard, the above results can be applied to nanoclass spacecraft with limited computing power. Recall that the Kotelnikov–Study transference principle [12] allows extending quaternionic formulas describing the control of angular motion to biquaternionic ones describing the control of the general spatial motion of a solid body. Based on the above results, this principle can be used to obtain an analytical quasi-optimal program control algorithm for the spatial motion (maneuvering) of a spacecraft.

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